Course: Operational Research

Field of study: Management and Production Engineering

Type of instruction and number of hours: lecture 15 h, laboratory 30 h

Number of ECTS credits: 3

Learning outcomes:

Knowledge:

- Student knows the areas of use of operational research, with particular emphasis on optimisation problems.
- Student knows typical optimisation methods supporting the implementation of projects and conditions for their application.
- Student can formulate a decision problem and build a mathematical model of the problem.

Skills:

- Student can choose methods and solve typical problems of logistics management and project planning.
- Student can use selected IT tools to solve optimisation problems.

Social competences:

- Student is aware of the importance of using quantitative methods in business management.
- Student is aware of the advantages and limitations of known methods for their practical use.

Evaluation methods of learning outcomes:
written test, activity during classes – solving tasks

List of course topics:

Lecture:

1. Operational research – essence, genesis, development.
2. Linear optimisation models.
3. Transportation problems.
4. The problem of maximum flow.
5. The travelling salesman problem.
6. Nonlinear optimisation models.
7. Network optimisation models and their applications.
8. Dynamic programming.
9. Inventory management.
10. Multi-criteria problems.

Laboratory:

1. Construction of linear optimisation models. Solving linear decision problems.
2. Building models for transportation problems and finding optimal solutions.
3. Selected network optimisation models.
4. Inventory control.
5. Selected nonlinear optimisation algorithms.
6. Multi-criteria optimisation.

Bibliography

Basic literature

[1] Eiselt H. A., Sandblom C.-L., *Linear Programming and Its Applications*. Springer, Berlin, Heidelberg, New York, 2007.

[2] Taha H. A., *Operations Research – An Introduction*. Pearson Prentice Hall, New Jersey, 2017.

Complementary literature

[3] Matoušek J., Gärtner B., *Understanding and Using Linear Programming*. Springer-Verlag, Berlin, Heidelberg, 2007.

[4] Antoniou A., Lu W.-S., *Practical Optimization – Algorithms and Engineering Applications*. 2nd Edition, Springer, New York, 2021.

Websites

[5] The Operational Research Society: https://www.theorsociety.com/about-or/

Decision making process

The decision-making process involves three major steps:
1) identifying the decision problem,
2) building a model of a decision situation,
3) decision making – choice of solution.

The construction of models that capture the complexity of decision problems in a logical framework is the essence of the approach appropriate for operations research. In most cases, operational research models are used to determine optimal solutions, therefore they are called optimisation models.

Linear programming

Linear programming deals with solving problems involving decision situations that can be described by a so-called linear programme, a model in which the constraint conditions and the objective function have a linear form. The steps in solving linear programming problems are the same as for all decision problems:
1) problem formulation,
2) creation of a mathematical model,
3) solving the problem using mathematical tools.

Building a mathematical model

To apply linear programming to decision making, a mathematical model must be developed. It consists of:
a) decision variables – describe the tools and resources available to achieve a goal; they take non-negative values,
b) conditional constraints – are constraints that may arise in the course of achieving our goal; constraints are presented in the form of equations, the left hand sides of which are linear forms,
c) objective function (criterion function) – illustrates the goal we want to achieve; it must be a linear function that depends on all decision variables.

Constraint conditions can take the form:

$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \leq a$

$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \geq a$

$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = a$

where:

$x_1, x_2, \ldots, x_n$ – decision variables the magnitudes of which we are looking for,

$a_1, a_2, \ldots, a_n$ – coefficients with variables (e.g. consumption of raw material per unit, content of an ingredient in a product per unit, etc.),

a – right-hand sides of constraints (e.g. limit of raw material to be used, minimum ingredient content in the mixture, etc.).

The objective function is a function of all decision variables and takes the form:

$$F(x_1, x_2, \ldots, x_n) = b + c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

where:

$c_1, c_2, \ldots, c_n$ – objective function coefficients (e.g. costs per unit, profits per unit, etc.).

The objective function is minimised (e.g. total purchase cost), maximised (e.g. profit or sales revenue), or may seek to achieve a specific value.

Solution

By solving a problem, we obtain a set of feasible solutions that will satisfy the constraints. If the model has two decision variables, it is possible to use the geometric method in Cartesian coordinate system to obtain a solution.

For problems with a larger number of variables, the simplex method is applicable. Due to the fact that the simplex method is quite time-consuming, there is a growing interest in computer packages that allow solving problems in the scope of operations research (among others, the Solver add-in for Microsoft Office Excel).

Linear programming problems can:

a) have a single optimal solution,
b) have more than one optimal solution, the so-called set of alternative optimal solutions, i.e. solutions for which the obtained values of variables are different, but the value of the objective function is the same (optimal from the point of view of the objective function),
c) have no solution in a situation of conflicting constraints (it is impossible to determine a single solution for which all the constraints would be satisfied).

Application of linear programming

Linear programming is applied in the case of the following typical decision problems:

1) product mix problems (e.g. maximising profit or revenue given constraints on resource availability or production capacity),
2) mixture problems (e.g., minimising the cost of purchasing raw materials to produce the final product under restrictions on its composition),
3) selection of technological processes (e.g. minimisation of waste costs under the constraints of accepted orders),
4) transport problems (e.g. minimising transport costs from suppliers to customers – a specific case of linear programming).

Example no. 1 of a task with solution

Task content

A company produces two products: P1 and P2. Two limited resources are consumed, R1 and R2, during the production process. The resource consumption per unit of each product, the allowable limits of raw material consumption and the selling prices of the products are provided in the table below.

Resources (raw materials)	Consumption per product unit [units]		Resources limits [units]
	P1	P2	
R1	8	12	4800
R2	8	5	4000
Unit price [€]	300	400	

In addition, it is known that the production capacity of the department that is the bottleneck of the production process does not allow to produce more than 360 units of P1 and 300 units of P2.

Determine the optimum production programme ensuring the maximum sales revenue under the existing constraints. Use the graphical method and MS Excel Solver add-in.

Building a mathematical model of a problem

First, it is necessary to define the decision variables. The task is to establish the optimal production programme to maximise sales revenue. Therefore, the production volume of particular products needs to be determined:
x_1 – production volume of product P1
x_2 – production volume of product P2

Next step is defining the constraints.
The first group of constraints concerns the limits on the consumption of raw materials. For resource R1, the consumption is 8 units per unit of product P1 and 12 units per unit of product P2. Therefore, the total consumption of material R1 in the production process can be written as: $8x_1 + 12x_2$. The consumption limit that cannot be exceeded is 4800 units. In this case we write the limit for the consumption of material R1 as:

$$(1)\ 8x_1 + 12x_2 \leq 4\,800$$

Similarly, we write the consumption limit for material R2 as:

$$(2)\ 8x_1 + 5x_2 \leq 4\,000$$

The second group of constraints concerns the production capacity of the process. It is known that the production capacity of the department that is the bottleneck of the production process does not allow to produce more than 360 units of product P1 (decision variable x_1) and 300 units of product P2 (decision variable x_2). In this case we write the limiting conditions in this case as:

$$(3)\ x_1 \leq 360$$
$$(4)\ x_2 \leq 300$$

Another type of constraint is a boundary condition, which is the range within which the decision variable can fall. Boundary conditions can restrict the values of variables from below as well as from above. In our case, we are dealing with non-negative variables (output cannot be less than 0), so:

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Constraints (3) and (4) are also actually boundary conditions (they specify the upper acceptable values of the decision variables).

In the next step, we determine the form of the objective function. In this task, it is necessary to determine the optimal production volume to ensure maximum sales revenue. Therefore, knowing the sales unit prices of individual products, the sales revenue can be written as $300x_1 + 400x_2$. The value of the objective function depends on the sales volume of both products, so it is a function of two variables. Since our objective is to maximise it, we write it as follows:

$$F(x_1, x_2) = 300x_1 + 400x_2 \rightarrow \max$$

The complete mathematical model (linear programming task) is as follows:

x_1 – production volume of product P1
x_2 – production volume of product P2

(1) $8x_1 + 12x_2 \leq 4\,800$
(2) $8x_1 + 5x_2 \leq 4\,000$

(3) $x_1 \leq 360$
(4) $x_2 \leq 300$

$x_1 \geq 0$
$x_2 \geq 0$

$F(x_1, x_2) = 300x_1 + 400x_2 \rightarrow \max$

Of course, the lower boundary conditions can be merged with constraints (3) and (4) (upper boundary conditions). Then, the model will be as follows:

x_1 – production volume of product P1
x_2 – production volume of product P2

(1) $8x_1 + 12x_2 \leq 4\,800$
(2) $8x_1 + 5x_2 \leq 4\,000$
(3) $0 \leq x_1 \leq 360$
(4) $0 \leq x_2 \leq 300$

$F(x_1, x_2) = 300x_1 + 400x_2 \rightarrow \max$

Clearly, both forms of the model notation are equivalent.

Solving the problem – graphical method

A graphical (geometric) method will be used to determine the optimal solution. This method is applicable to tasks with two decision variables.
The boundary conditions – non-negative variables – limit the range of solutions to the first quadrant of the coordinate system. The axes of the system correspond to the decision variables.
We begin the solution by determining the set of feasible solutions due to the constraints. To this end, we present the constraints in a graphical form.

We start with condition (1): $8x_1 + 12x_2 \leq 4\,800$. To present the equation $8x_1 + 12x_2 = 4\,800$ associated to the constraint (1) graphically, we need to find 2 points through which this straight line passes. The easiest way to do this is to find the points of intersection of this straight line with the axes of the coordinate system.

Assuming $x_1 = 0$, we get $x_2 = 400$ (4 800/12).
Similarly, assuming $x_2 = 0$, we get $x_1 = 600$ (4 800/8).

Thus, our line intersects the axes of the coordinate system at the points (0; 400) and (600; 0), where the first coordinate corresponds to the variable x_1 and the second one to the variable x_2.

All points on the segment bounded by points (0; 400) and (600; 0) denote the production volumes of products P1 and P2 that fully utilise raw material R1, to which the first limiting condition applies (Fig. 1).

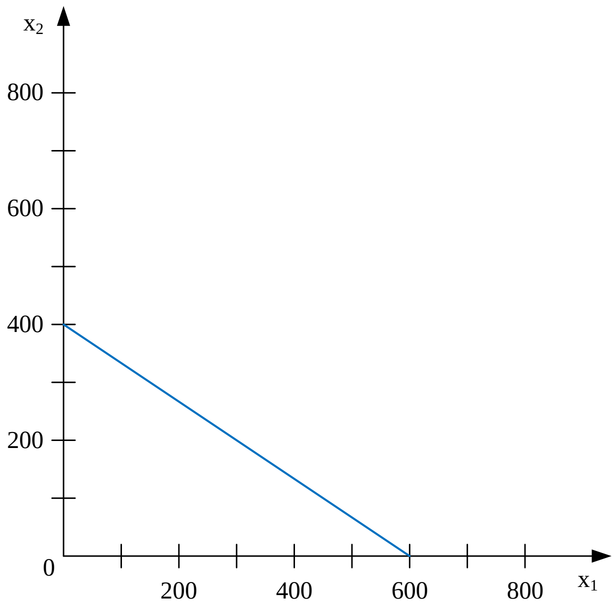


Figure 1.

The equation $8x_1 + 12x_2 \leq 4\,800$ is satisfied by all points located on and below the determined line. Considering the condition of non-negativity of variables, the marked area includes all feasible solutions due to the constraint (1). In other words, every solution from the marked area is possible concerning the availability of material R1 (Fig. 2).

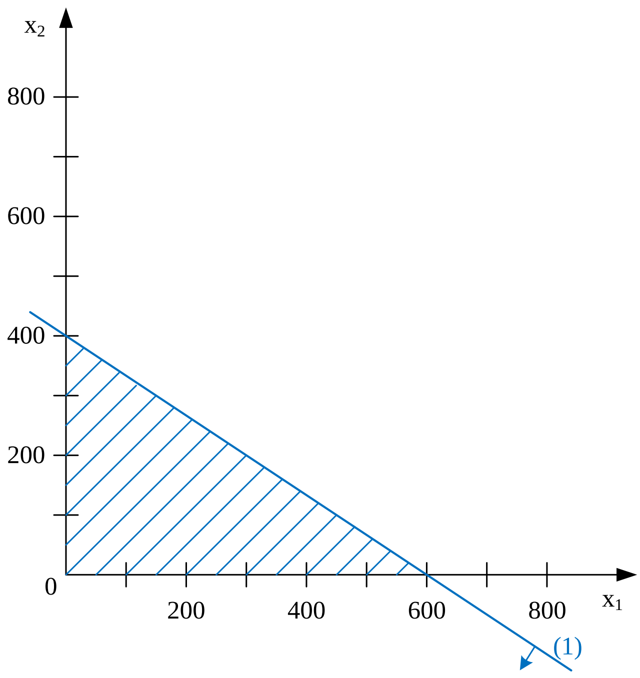


Figure 2.

In a similar way, we determine the graphical representation of condition (2): $8x_1 + 5x_2 \leq 4\,000$. We determine the points of intersection of the straight line $8x_1 + 5x_2 = 4\,000$ with the axes of the coordinate system.
Assuming $x_1 = 0$, we obtain $x_2 = 800$ (4 000/5).
On the other hand, assuming $x_2 = 0$, we get $x_1 = 500$ (4 000/8).

Thus, the line (2) intersects the axes of the coordinate system at the points (0; 800) and (500; 0).

Condition (2) is satisfied by all the points located on the segment bounded by the aforementioned points and below (taking into account the non-negativity of the decision variables).

Condition (3) $x_1 \leq 360$ is satisfied by all points located on and to the left of the vertical line $x_1 = 360$. While condition (4) $x_2 \leq 300$ is satisfied by all points located on the horizontal line $x_2 = 300$ and below.
In both cases, we keep in mind the non-negativity of the decision variables (first quadrant of the coordinate system).

After marking all the limiting conditions we determine the set of feasible solutions, i.e. the ones in the case of which all the constraints are satisfied (the hatched area) (Fig. 3). In the feasible polyhedron we will search for the optimal solution.
The feasible polyhedron is bounded by the vertexes A, B, C, D, and (0; 0). Any of these points will be an optimal solution. We may also have a set of optimal alternatives defined by a segment bounded by a pair of points located on any of the lines belonging to the bounding conditions.

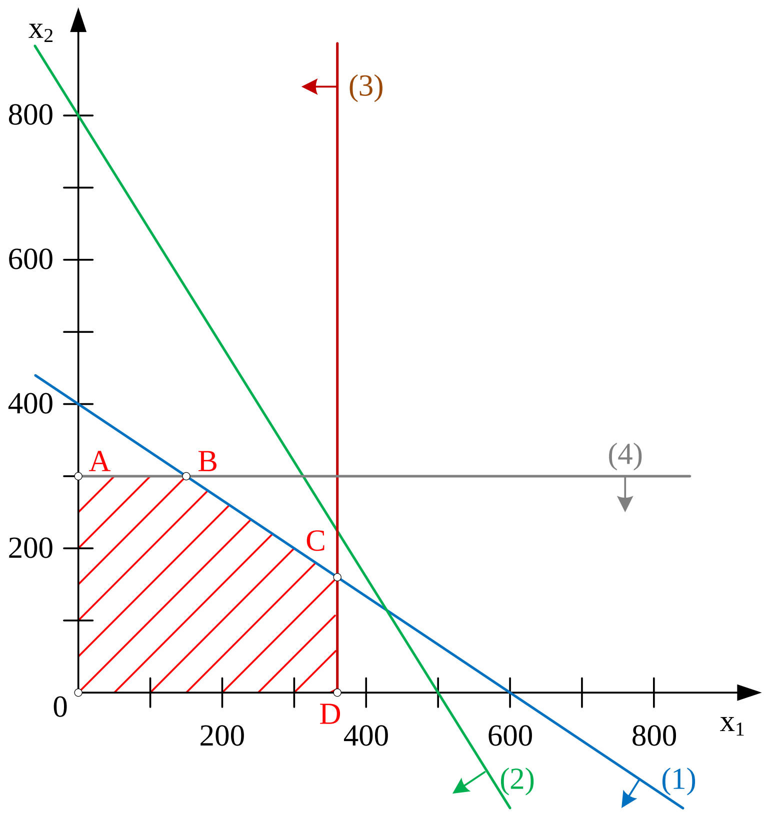


Figure 3.

In order to find the optimal solution, we determine the coordinates of the points indicated on the graph.

The coordinates of the points A and D can be read from the graph on the basis of the constraints (3) and (4). As a result, we obtain the following:
A (0; 300),
D (360; 0).

The point B is determined by the intersection of the lines associated to the constraints (1) and (4). So its coordinates are determined by solving the system of equations:

$$\begin{cases} 8x_1 + 12x_2 = 4800 \\ x_2 = 300 \end{cases}$$

After solving it, we obtain the following:

$$\begin{cases} x_1 = 150 \\ x_2 = 300 \end{cases}$$

B (150; 300).

The point C is determined by the intersection of the lines associated to the constraints (1) and (3), so its coordinates are determined by solving the system of equations:

$$\begin{cases} 8x_1 + 12x_2 = 4\,800 \\ x_1 = 360 \end{cases}$$

After solving it, we obtain the following:

$$\begin{cases} x_1 = 360 \\ x_2 = 160 \end{cases}$$

C (360; 160).

Next, we check the value of the objective function for each point. To do this, we substitute their coordinates for x_1 and x_2.

$$F(x_1, x_2) = 300x_1 + 400x_2 \rightarrow \max$$

A (0; 300); F(A) = 300 × 0 + 400 × 300 = 120 000
B (150; 300); F(B) = 300 × 150 + 400 × 300 = 165 000
C (360; 160); F(C) = 300 × 360 + 400 × 160 = 172 000
D (360; 0); F(D) = 300 × 360 + 400 × 0 = 108 000

For obvious reasons, we do not check the value of the objective function for the point (0; 0), since this point represents the production volume of 0 in the case of both products.
The objective function reaches its maximum value in the case of the point C, so the optimal solution is:

$$\begin{cases} x_1 = 360 \\ x_2 = 160 \end{cases}$$

That is, the production of 360 pieces or units of product P1 and 160 pieces or units of product P2. The value of the objective function, i.e. the maximum sales revenue, is € 172 000.
It is worth noting that an optimal solution is to be understood as the values of the decision variables, not the value of the objective function.

By checking the value of the objective function for each point, it was possible to limit the scope to the points B and C, omitting the points A and D.
By comparing the points A and B, it becomes clear that with positive coefficients of the objective function (sales unit prices of both products), the point B presents a better solution. In both points, the sale of product P2 is the same (300 units), but the point B additionally indicates that we sell 150 more units of product P1 (with 0 units of P1 in the point A).
Similarly, by comparing the points C and D, we see that in both of them the sale of product P1 is the same (360 units), but the point C shows that we sell 160 more units of P2 (with 0 units of P2 as seen in the point D).
Another way to determine the optimal solution is to plot the objective function on the graph. Let's choose the constant value in the equation of the objective function randomly. It is most convenient

to assume that any initial value of the objective function is the multiple of the objective function coefficients (products unit prices). The sales prices of products P1 and P2 are € 300 and € 400 respectively, so we may assume a sales revenue value of € 120 000:

$$F(x_1, x_2) = 300x_1 + 400x_2 = 120\,000$$

Next, we plot the straight line representing the given value of the objective function on the graph. The easiest way is to find the points of intersection of this line with the axes of the coordinate system.

Assuming $x_1 = 0$, we get $x_2 = 300$ (120 000/400).
Assuming $x_2 = 0$, we get $x_1 = 400$ (120 000/300).

Our objective function line, which is related to the sales revenue value of € 120 000, intersects the axes of the coordinate system at the points (0; 300) and (400; 0).
An optimum point usually is located on one of the corners of the feasible region (we may also have a set of optimal alternatives defined by a segment bounded by a pair of points located on any of the lines belonging to the feasible polyhedron). To find it, we have to place a ruler on the graph sheet, parallel to the objective function:

- if the goal is to maximise the objective function, we need to find the point of contact of the ruler with the feasible region, which is the farthest from the origin – this is the optimum point for maximising the function;
- if the goal is to minimise the objective function, we need to find the point of contact of the ruler with the feasible region, which is the closest to the origin – this is the optimum point for minimising the function.

The line I (Fig. 4) denotes the set of points (sales volume of products P1 and P2) in the case of which the sales revenue is equal to € 120 000. It crosses the region of feasible solutions, so we can see that by moving it upwards in parallel, it is possible to obtain a more favourable result.

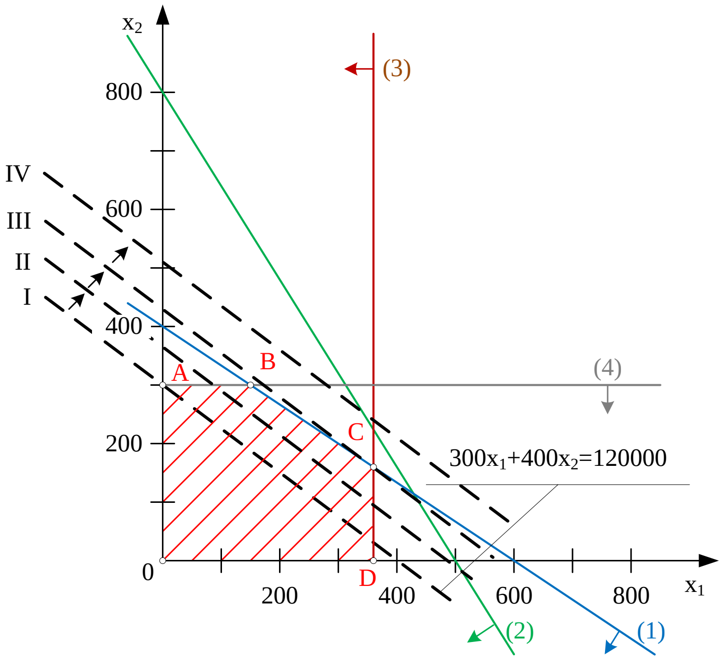


Figure 4.

The line II, like the line I, crosses the feasible region and it is possible for it to move further upwards, so it does not correspond to the maximum revenue. The line III passes through the point C and its further movement upwards (e.g. the line IV) results in going beyond the feasible polyhedron. Thus, the point C (360; 160) is the optimal solution for which the objective function is:

$$F(C) = 300\times360 + 400\times160 = 172\ 000.$$

The determination of the coordinates of this point is described in the explanation of the previous method of determining the optimum.

Solving the problem – MS Excel Solver

Figure 5. shows a proposed spreadsheet created for the task.

	A	B	C	D	E	F
1	Decision variables	x_1	x_2			
2				LHS of constraints		RHS of constraints
3	Constraint (1) - resource R1	8	12	=B3*B2+C3*C2	≤	4800
4	Constraint (2) - resource R2	8	5	=B4*B2+C4*C2	≤	4000
5	Constraint (3) - capacity P1	1		=B5*B2+C5*C2	≤	360
6	Constraint (4) - capacity P2		1	=B6*B2+C6*C2	≤	300
7	Objective function	300	400	=B7*B2+C7*C2	→	max

Figure 5.

Columns B and C contain the coefficients of the decision variables in the constraints and objective function. The gray cells B2 and C2 are the boxes where the optimal solution will be determined. In the column 'LHS of constraints' formulas have been created for the left hand sides of constraints, while in the column 'RHS of constraints' (right hand sides of constraints) resource and capacity constraints have been entered. The gray cell D7 contains the formula that will give the value of the objective function for the determined solution.

After preparing your worksheet, choose *Data → Solver* (if Solver is not available, we have to activate it: *File → Options → Add-ins → Manage Excel Add-ins → Go → Solver Add-in*). The completed window of the Solver add-in (including: the defined objective function, constraints of boundary conditions – non-negative variables, and selected solving method – Linear Programming Simplex) is shown in Figure 6.

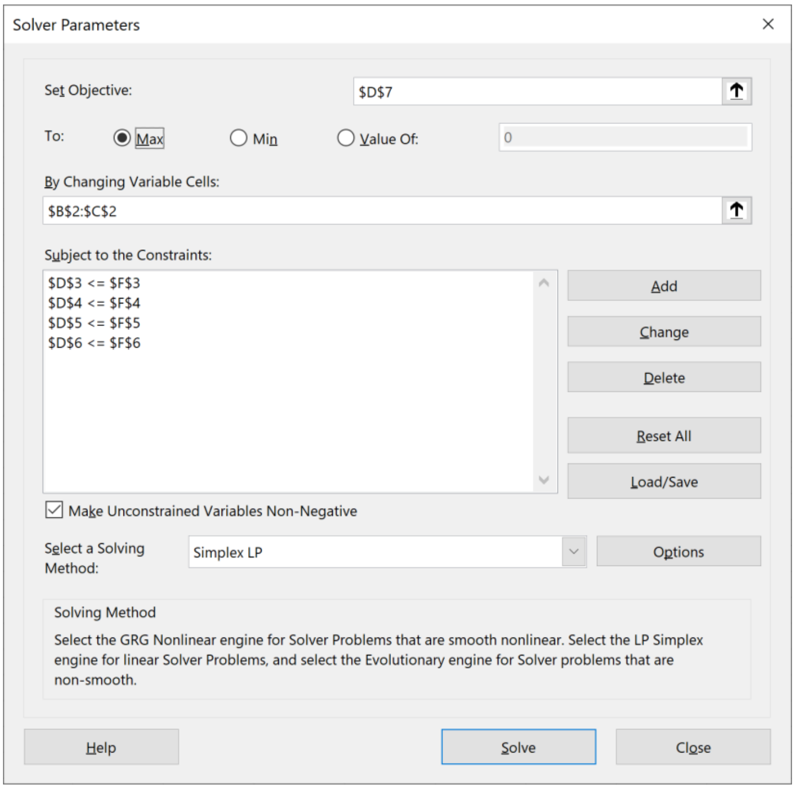


Figure 6.

Constraints are added by pressing *Add* and filling in another window (Fig. 7).

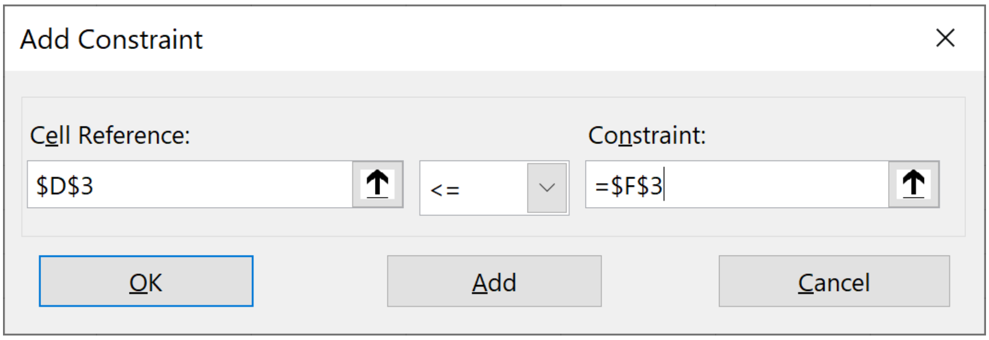
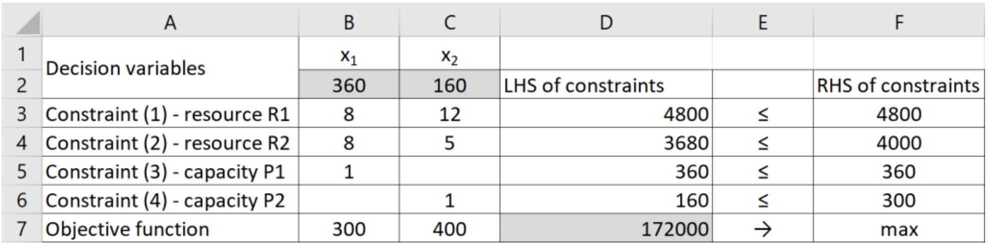


Figure 7.

After pressing *Solve,* we can select the available reports (Answers, Sensitivity, Limits) and finally obtain the optimal solution (Fig. 8).



	A	B	C	D	E	F
1	Decision variables	x_1	x_2			
2		360	160	LHS of constraints		RHS of constraints
3	Constraint (1) - resource R1	8	12	4800	≤	4800
4	Constraint (2) - resource R2	8	5	3680	≤	4000
5	Constraint (3) - capacity P1	1		360	≤	360
6	Constraint (4) - capacity P2		1	160	≤	300
7	Objective function	300	400	172000	→	max

Figure 8.

As it can be seen, the optimal solution obtained using Solver add-in coincides with the solution obtained through the graphical method:

$$\begin{cases} x_1 = 360 \\ x_2 = 160 \end{cases}$$

The value of the objective function (maximum sales revenue) is € 172 000.

Example no. 2 of a task with solution

Task content

A company produces two products: P1 and P2. Two limited resources are consumed, R1 and R2, during the production process. The resource consumption per unit of each product, the allowable limits of raw material consumption and the profit per unit of each product are provided in the table below.

Resources (raw materials)	Consumption per product unit [units]		Resources limits [units]
	P1	P2	
R1	5	5	3 000
R2	8	4	4 000
Unit price [€]	20	20	

In addition, it is known that the production capacity of the department that is the bottleneck of the production process does not allow to produce more than 400 units of P2.

Determine the optimum production programme ensuring the maximum sales revenue under the existing constraints. Use the graphical method.

A mathematical model of the problem

The task is to establish the optimal production programme to maximise sales profit. Therefore, the production volume of particular products needs to be determined. The complete mathematical model (linear programming task) is as follows:

x_1 – production volume of product P1
x_2 – production volume of product P2

(1) $5x_1 + 5x_2 \le 3\,000$
(2) $8x_1 + 4x_2 \le 4\,000$
(3) $x_2 \le 400$

$x_1 \ge 0$
$x_2 \ge 0$

$F(x_1, x_2) = 20x_1 + 20x_2 \rightarrow \max$

The first two constraints refer to limited resources, the third one to limited production capacity.

Solving the problem

The graphical representation of the model was developed similarly to the Example no. 1.
The equation $5x_1 + 5x_2 \le 3000$ is satisfied by all points located on and below the line that intersects the axes of the coordinate system at the points (0; 600) and (600; 0). Similarly, the equation $8x_1 + 4x_2 \le 4\,000$ is satisfied by all points located on and below the line that intersects the axes of the coordinate system at the points (0; 1 000) and (500; 0). Condition (3) $x_2 \le 400$ is satisfied by all points located on the horizontal line $x_2 = 400$ and below. In all cases, we keep in mind the non-negativity of the decision variables (first quadrant of the coordinate system).
After marking all the limiting conditions we determine the set of feasible solutions, i.e. the ones in the case of which all the constraints are satisfied (the hatched area) (fig. 9). In the feasible polyhedron we will search for the optimal solution.
The feasible polyhedron is bounded by the vertexes A, B, C, D, and (0; 0). Any of these points will be an optimal solution. We may also have a set of optimal alternatives defined by a segment bounded by a pair of points located on any of the lines belonging to the bounding conditions.

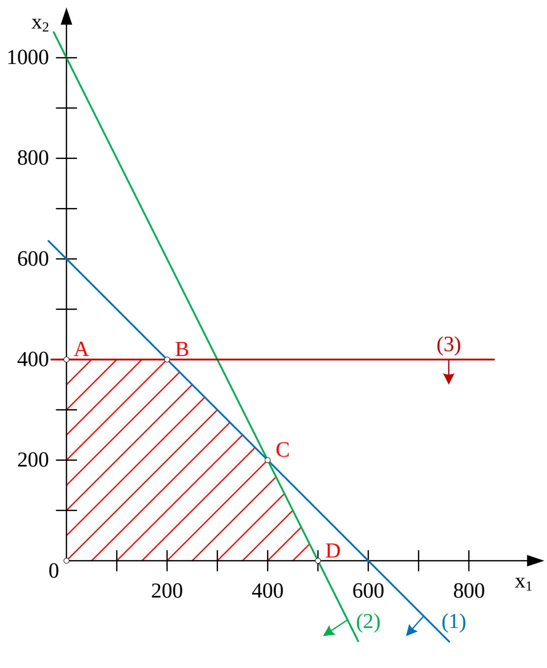


Figure 9.

In order to find the optimal solution, we determine the coordinates of the points indicated on the graph.

The coordinates of the points A and D can be read from the graph on the basis of the constraint (3):
A (0; 400),
D (500; 0).

The point B is determined by the intersection of the lines associated to the constraints (1) and (3). So its coordinates are determined by solving the system of equations:

$$\begin{cases} 5x_1 + 5x_2 = 3\,000 \\ x_2 = 400 \end{cases}$$

After solving it, we obtain the following:

$$\begin{cases} x_1 = 200 \\ x_2 = 400 \end{cases}$$

B (200; 400).
The point C is determined by the intersection of the lines associated to the constraints (1) and (2), so its coordinates are determined by solving the system of equations:

$$\begin{cases} 5x_1 + 5x_2 = 3\,000 \\ 8x_1 + 4x_2 = 4\,000 \end{cases}$$

After solving it, we obtain the following:

$$\begin{cases} x_1 = 400 \\ x_2 = 200 \end{cases}$$

C (400; 200).

Next, we check the value of the objective function for each point. To do this, we substitute their coordinates for x_1 and x_2.

$$F(x_1, x_2) = 20x_1 + 20x_2 \rightarrow \max$$

A (0; 400); F(A) = 20 × 0 + 20 × 400 = 8 000
B (200; 400); F(B) = 20 × 200 + 20 × 400 = 12 000
C (400; 200); F(C) = 20 × 400 + 20 × 200 = 12 000
D (500; 0); F(D) = 20 × 500 + 20 × 0 = 10 000

For obvious reasons, we do not check the value of the objective function for the point (0; 0), since this point represents the production volume of 0 in the case of both products.
The objective function reaches its maximum value in the case of the points B (200; 400) and C (400; 200). Given that we are dealing with linear programming (all constraints and the objective function have linear form), not only the points B and C are optimal solutions. All points lying on the segment bounded by the points B and C are the set of alternative optimal solutions. From the company's point of view, the sale defined by the coordinates of each of these points is equally profitable – for each of these points the value of the objective function (total profit) is 12000 and is maximum.

To check, sample points E (250; 350) and F (300; 300) are plotted on the segment connecting points B and C (Fig. 10). After substituting their coordinates into the objective function, we obtain:
F(E) = 20 × 250 + 20 × 350 = 12 000
F(F) = 20 × 300 + 20 × 300 = 12 000

As it is seen, these are solutions that give the same value of profit as the points B and C.

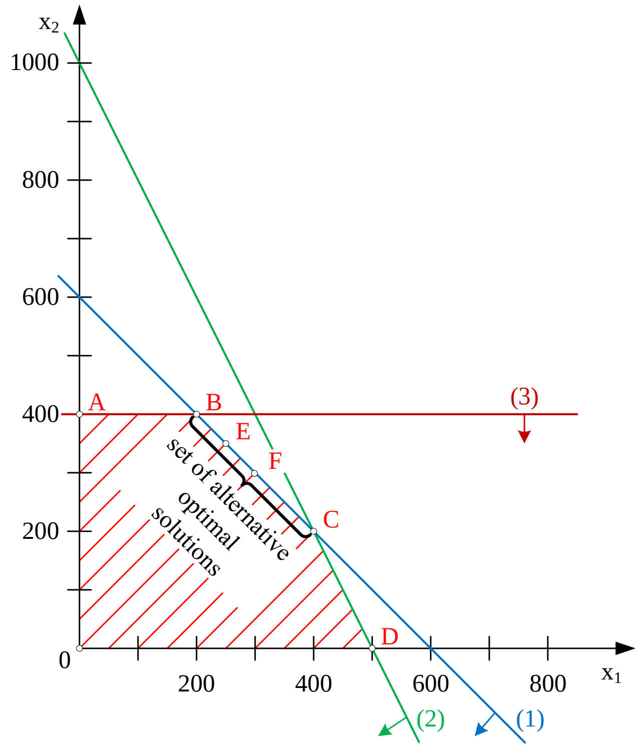


Figure 10.

Another way to determine the optimal solution is to plot the objective function on the graph. The profits per unit of products P1 and P2 are € 20 and € 20 respectively, so we may assume a sales revenue value of € 4 000:

$$F(x_1, x_2) = 20x_1 + 20x_2 = 4\ 000$$

Next, we plot the straight line representing the given value of the objective function on the graph. The easiest way is to find the points of intersection of this line with the axes of the coordinate system.

Assuming $x_1 = 0$, we get $x_2 = 200$ (4 000/20).
Assuming $x_2 = 0$, we get $x_1 = 200$ (4 000/20).

Our line I of total profit of € 4 000 intersects the axes of the coordinate system at the points (0; 200) and (200; 0) (Fig. 11).

The line I (Fig. 11) denotes the set of points (sales volume of products P1 and P2) in the case of which the sales profit is equal to € 4 000. It crosses the region of feasible solutions, so we can see that by moving it upwards in parallel, it is possible to obtain a more favourable result.
The lines II and III, like the line I, cross the feasible region and it is possible for them to move further upwards, so they do not correspond to the maximum profit. The line IV passes through the points B and C – it coincides with line (1) – and its further movement upwards (e.g. the line V) results in going beyond the feasible polyhedron. Thus, all points on the segment bounded by the points B and C are the set of alternative optimal solutions providing maximum profit of € 12 000. The determination of the coordinates of the points B and C and the values of the objective function for these points are described in the explanation of the previous method of determining the optimum.

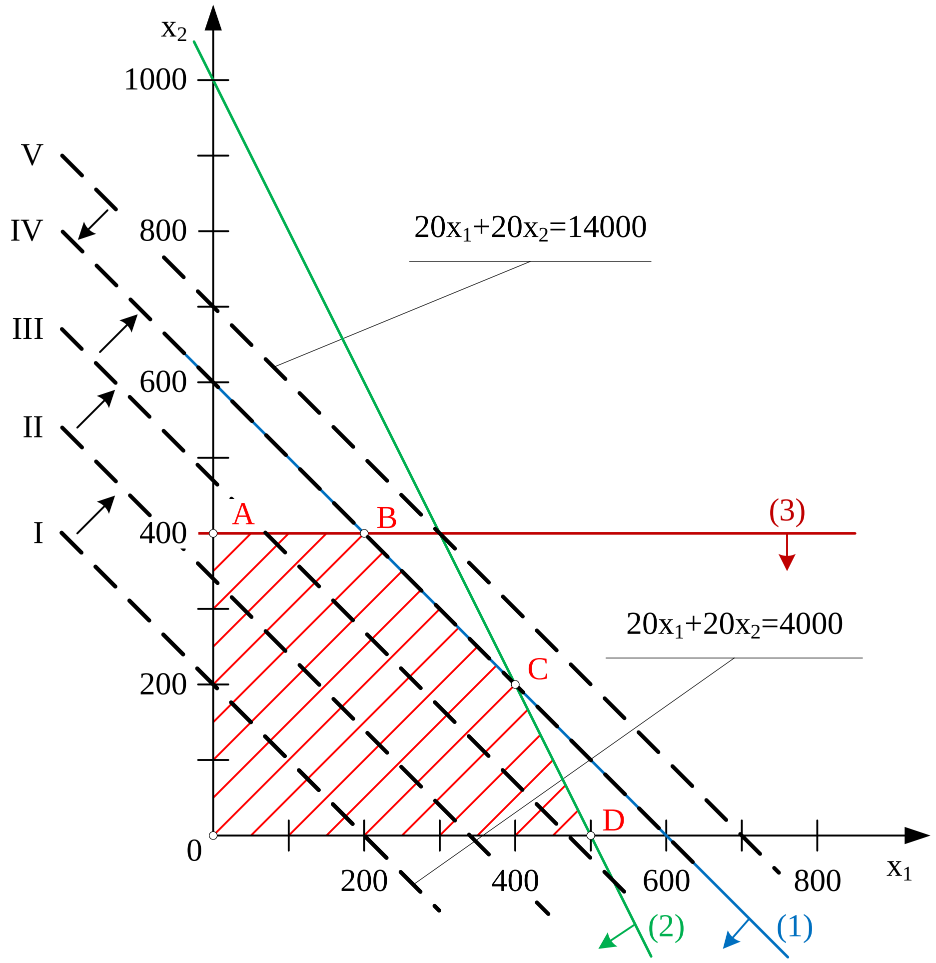


Figure 11.

Example no. 3 of a task with solution

Task content

The farm raises pigs. The animals must be provided with nutrient N1 of at least 1 000 units and nutrient N2 of at least 400 units. The farmer is considering purchasing two products: P1 and P2. These products also contain nutrients N3 and N4, which cannot be supplied in amounts greater than 800 and 600 units, respectively, due to harmful effects. The nutrient contents of each product and the purchase prices of the products are included in the table below.

Nutrients	Nutrient content per product unit [units]		Minimum nutrient content [units]	Maximum nutrient content [units]
	P1	P2		
N1	50	50	1 000	–
N2	15	45	450	–
N3	80	100	–	800
N4	40	120	–	600
Unit price [€]	50	60		

Determine the purchase volume of products P1 and P2 to meet the feed composition requirements at the minimum purchase cost.

A mathematical model of the problem

The task is to determine a purchase plan for animal feed products that provides the required composition at the minimum purchase cost. Therefore, the purchase volume of particular products needs to be determined. The complete mathematical model (linear programming task) is as follows:

x_1 – purchase volume of product P1 (number of units)
x_2 – purchase volume of product P2 (number of units)

(1) $50x_1 + 50x_2 \geq 1000$
(2) $15x_1 + 45x_2 \geq 450$
(3) $80x_1 + 100x_2 \leq 800$
(4) $40x_1 + 120x_2 \leq 600$

$x_1 \geq 0$
$x_2 \geq 0$

$F(x_1, x_2) = 50x_1 + 60x_2 \rightarrow \min$

The first two constraints relate to the minimum content of nutrients N1 and N2, while the next two constraints relate to the maximum content of nutrients N3 and N4.

Solving the problem

The graphical representation of the model was developed similarly to the Examples no. 1 and no. 2.

The equation $50x_1 + 50x_2 \geq 1000$ is satisfied by all points located on and above the line that intersects the axes of the coordinate system at the points (0; 20) and (20; 0). Similarly, the equation $15x_1 + 45x_2 \geq 450$ is satisfied by all points located on and above the line that intersects the axes of the coordinate system at the points (0; 10) and (30; 0). The equation $80x_1 + 100x_2 \leq 800$ is satisfied by all points located on and below the line that intersects the axes of the coordinate system at the points (0; 8) and (10; 0). And finally, the equation $40x_1 + 120x_2 \leq 600$ is satisfied by all points located on and below the line that intersects the axes of the coordinate system at the points (0; 5) and (15; 0). In all cases, we keep in mind the non-negativity of the decision variables (first quadrant of the coordinate system).

When all the constraints are plotted, it can be seen that there is not a single point that satisfies all the constraints (fig. 12). There is an area satisfying constraints (1) and (2) on the minimum content of nutrients N1 and N2 (hatched area A) and an area satisfying constraints (3) and (4) on the maximum content of nutrients N3 and N4 (hatched area B), but there is no feasible with at least one solution satisfying all the constraints.

Thus, in this case we are dealing with a linear programming task that has no solution − there is no feasible.

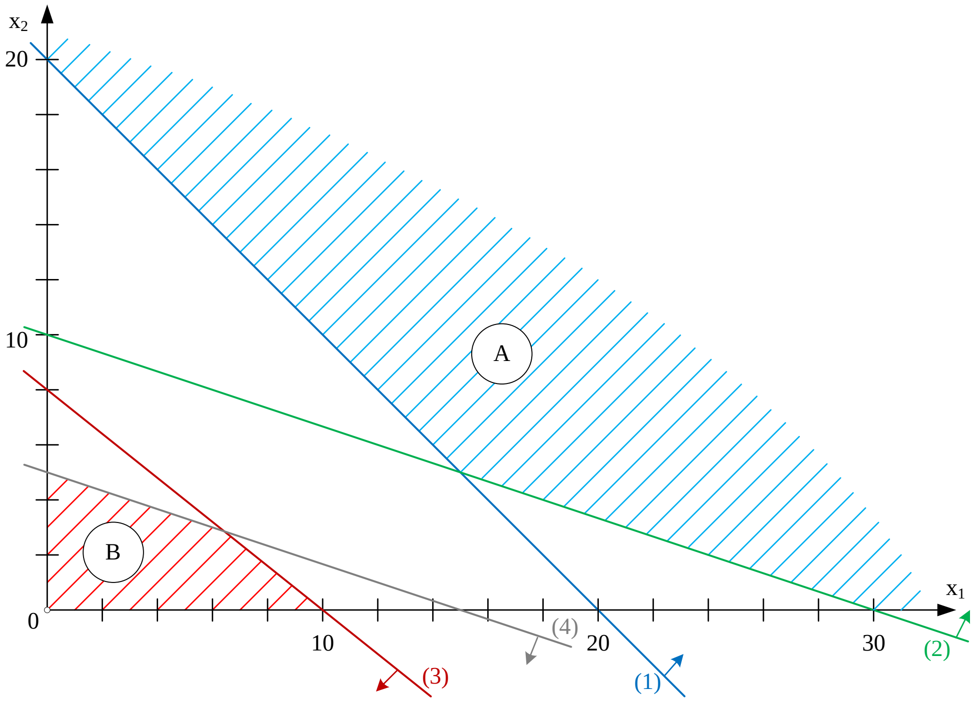


Figure 12.

Control tasks

1. A company X makes 2 products: P1, P2. Three manufacturing resources are limited: limit for resource R1 is 500 units per month, R2 – 50 units per month, and R3 – 200 units per month. The enterprise uses 5 units of R1, 1 unit of R2, and 4 units of R3 to produce a unit of P1. 10 units of R1, 2 units of R2, and 2 units of R3 are consumed to produce a unit of P2. The profit per unit in the case of P1 is € 20 and € 25 for P2.
   How many units of P1 and P2 should be made monthly to maximise the profit? Calculate the profit for the optimal solution. Formulate the decision problem and solve it using the graphical method as well as MS Excel add-in.

2. A special supplement for athletes is a mixture of two commercially available products: P1 and P2. The mixture to be produced should contain at least 70 mg, but no more than 180 mg, of vitamin A, at least 40 mg of vitamin C, and at least 60 mg of vitamin D. 1 kg of product P1 costs € 5 and contains 2 mg of vitamin A, 2 mg of vitamin C, and 6 mg of vitamin D. 1 kg of product P2 costs € 8 and contains 9 mg of vitamin A, 4 mg of vitamin C and 2 mg of vitamin D. Determine quantities of products that should be mixed in order to obtain the cheapest possible supplement containing the necessary amounts of vitamins. Formulate the decision problem and solve it using the graphical method as well as MS Excel add-in.

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