Course: **Financial Mathematics**

Field of Study: Finance and Accounting

Form of classes and number of hours: lecture 15 h, practical classes 15 h

Number of ECTS credits: 4

Learning outcomes:

- Student is familiar with the concept of the time value of money.
- He knows the idea: annuity, credit, capitalisation.
- Student is able to calculate present and future value of money using different methods of capitalisation.
- Student is able to calculate the value of present and future annuity and perpetuity.
- He can set up a variety of debt repayment plans, compute the amount of payments, amount of interest installment.
- He can evaluate debt instruments.

Evaluation methods of learning outcomes:
written exam

Subject matter of the classes:

1. Time value of money
2. Annuity
3. Credits
4. Valuation of debt instruments

References

Books

[1] Eugene F. Brigham, Joel F. Houston; *Fundamentals of financial management*; The Dryden Press Series in Finance, 15th edition Cengage Learning, 2019.

[2] James R. McGuigan, William J. Kretlow, R. Charles Moyer; *Contemporary Corporate Finance*; South-Western Cengage Learning, 2009.

[3] Eugene F. Brigham and Phillip R. Daves, *Intermediate Financial Management*, 14th edition, Cengage Learning, 2021.

[4] Eugene F. Brigham and Michael C. Ehrhardt, *Financial Management*, 15th edition, Cengage Learning, 2016.

[5] Eugene F. Brigham, Michael C. Ehrhardt; *Financial management: Theory and Practice*; South-Western Cengage Learning, 2014.

Websites

[6] https://corporatefinanceinstitute.com/resources/knowledge/valuation/time-value-of-money/

[7] Dr. Sharon Garrison; University of Arizona; www.studyfinance.com

Table of content

Financial mathematics

Financial mathematics – a field of applied mathematics. It describes the richness of economic reality with the help of a mathematical apparatus.

Today, it is believed that the foundations for the emergence of the field of financial mathematics were laid with the publication of a doctoral dissertation by the French mathematician and economist Louis Bachelier. However, his ideas remained forgotten for a long time and the rapid development of this field of knowledge did not take place until the 1970s, when Robert Merton and Myron Scholes introduced mathematical models of arbitrage valuation of financial instruments. This resulted, inter alia, in awarding them the Nobel Prize in 1997.

Financial mathematics is divided into two main strands:

- The theory of the function of money in time – uses basic knowledge in the field of mathematical analysis. It examines how capital changes over time, assuming a given interest rate. In addition, it considers payment sequences made every certain period.
- The theory of valuation of financial instruments – in addition to the knowledge of mathematical analysis, concepts related to the theory of probability are used in it. In particular, this theory is used in the valuation of stocks, bonds, and derivatives

1. Time value of money

The time value of money is a basic financial concept where money in the present is worth more than the same amount of money that could be received in the future. This is true because the money you have at the moment can be invested and you can get a return, thus creating a larger amount of money in the future. With future money, there is an additional risk that the money may never actually be received, for one reason or another. The time value of money is an important concept not only for individuals but also for making business decisions. Firms consider the time value of money when deciding to invest in new product development, acquire new equipment or business facilities, and establish credit terms to sell their products or services.

1.1. Future value of money

Symbols:

r – nominal interest rate (annual rate),

PV – start-up capital,

FV_n – capital value after n- years,

How to calculate future value of money?

After one year we have:

$$FV_1 = PV + r \cdot PV = PV(1+r)$$

After the second year:

$$FV = FV_1 + r \cdot FV = FV_1(1+r) = PV(1+r)^2$$

After the third year:

$$FV_3 = FV_2 + r \cdot FV_2 = FV_2(1+r) = PV(1+r)^3$$

Thus after n– years:

$$FV_n = PV(1+r)^n$$

If we denote the number of capitalisations in a year by m, the formula takes the following form:

$$FV_n = PV\left(1+\frac{r}{m}\right)^{m\cdot n}$$

If capitalisation is continuous, then we have to find the limit (m goes to infinity)

$$FV_n = \lim_{m\to\infty} PV\left(1+\frac{r}{m}\right)^{m\cdot n} = \lim_{m\to\infty} PV\left[\left(1+\frac{r}{m}\right)^{m}\right]^n = PVe^{rn}$$

Factor $\left(1+\frac{r}{m}\right)^{m\cdot n}$ is called the future value multiplier and is denoted by

$$FVM\left(\frac{r}{m}, m\cdot n\right)$$

Example 1.

Determine the future value of PLN 1 000 in 4 years, with a nominal rate of r = 12%,
if the capitalisation is:

a) Annual,

b) Semi-annual,

c) Quarterly,

d) Monthly,

e) Continuous.

Solution:

a) $m = 1, FV_4 = 1\,000(1 + 0.12)^{4=}1\,573.52$

b) $m = 2, FV_4 = 1\,000\left(1 + \frac{0.12}{2}\right)^8 = 1\,593.85$

c) $m = 4, FV_4 = 1\,000\left(1 + \frac{0.12}{4}\right)^{16} = 1\,604.71$

d) $m = 4, FV_4 = 1\,000\left(1 + \frac{0.12}{12}\right)^{48} = 1\,612.23$

e) $m = \infty, FV_4 = 1\,000e^{0.12\cdot 4} = 1\,616.07$

1.2. Present value of money

We transform the formula and

$$PV = \frac{FV_n}{\left(1 + \frac{r}{m}\right)^{m\cdot n}}$$

Example 2.

How much money should be paid to get PLN 1 000 after 3 years, with a nominal rate of r = 10%, if the capitalisation is:

a) Annual,

b) Semi-annually.

Solution

a) m = 1, $PV = \frac{1000}{(1+0.1)^3} = 751.31$

b) m = 2, $PV = \frac{1000}{\left(1+\frac{0.1}{2}\right)^6} = 746.22$

For continuous capitalisation we have:

$$PV = \frac{FV_n}{e^{rn}} = FV_n \cdot e^{-rn}$$

Factor $\frac{1}{\left(1+\frac{r}{m}\right)^{m\cdot n}}$ is called the present value multiplier and is denoted by

$$PVM\left(\frac{r}{m}, m \cdot n\right)$$

1.3. Nominal interest rate

When we want to calculate nominal interest rate, first we need to transform the formula for the future value of money

$$FV_n = PV\left(1+\frac{r}{m}\right)^{m\cdot n}$$

$$\frac{FV_n}{PV} = \left(1+\frac{r}{m}\right)^{m\cdot n}$$

$$\sqrt[m\cdot n]{\frac{FV_n}{PV}} = 1+\frac{r}{m}$$

$$\sqrt[m\cdot n]{\frac{FV_n}{PV}} - 1 = \frac{r}{m}$$

$$r = m\left(\sqrt[m\cdot n]{\frac{FV_n}{PV}} - 1\right)$$

Example 3.

What should the nominal interest rate be for the capital to double in value after 10 years, if the capitalisation is:

a) Annual,
b) Semi-annual,
c) Quarterly.

Solution

a) $m = 1, r = \left(\sqrt[10]{\frac{2PV}{PV}} - 1\right) = \sqrt[10]{2} - 1 = 0{,}071 = 7.1\%$

b) $m = 2, r = 2\left(\sqrt[20]{\frac{2PV}{PV}} - 1\right) = 2\left(\sqrt[20]{2} - 1\right) = 0.070$ = 7%

c) $m = 12, r = 12\left(\sqrt[120]{\frac{2PV}{PV}} - 1\right) = 12\left(\sqrt[120]{2} - 1\right) = 0.069 = 6.9\%$

For continuous capitalisation, we have

$$FV_n = PVe^{rn}$$

$$\frac{FV_n}{PV} = e^{rn}$$

$$rn = ln\frac{FV_n}{PV}$$

$$r = \frac{1}{n}ln\frac{FV_n}{PV}$$

1.4. Number of years of investment

First we need to transform the formula

$$FV_n = PV\left(1+\frac{r}{m}\right)^{m\cdot n}$$

$$\frac{FV_n}{PV} = \left(1+\frac{r}{m}\right)^{m\cdot n}$$

$$m\cdot n = log_{\left(1+\frac{r}{m}\right)}\frac{FV_n}{PV} = \frac{log\frac{FV_n}{PV}}{log\left(1+\frac{r}{m}\right)}$$

$$n = \frac{1}{m}\frac{log\frac{FV_n}{PV}}{log\left(1+\frac{r}{m}\right)}$$

Example 4.

For how many years should PLN 1 000 be paid to earn PLN 200 in interest, with a nominal interest rate r = 8%, if the capitalisation is:

a) Annual,
b) Semi-annual,
c) Quarterly.

Solution

a) $m = 1,\ n = \frac{log\frac{1200}{1000}}{log(1+0.08)} = 2.37$ Rep. 3 years

b) $m = 2,\ n = \frac{1}{2}\frac{log\frac{1200}{1000}}{log(1+0.04)} = 2.32$ Rep. 2.5 years

c) $m = 4,\ n = \frac{1}{4}\frac{log\frac{1200}{1000}}{log(1+0.02)} = 2.30$ Rep. 2.5 years (2 years and 2 quarters)

For continuous capitalisation, we have

$$FV_n = PVe^{rn}$$

$$\frac{FV_n}{PV} = e^{rn}$$

$$rn = ln\frac{FV_n}{PV}$$

$$n = \frac{1}{r}ln\frac{FV_n}{PV}$$

2. Annuity

An annuity is a sequence of payments at regular intervals. The time between payments is called the payment period. We will show it in the example below.

At the beginning of each quarter we make payments of PLN 1 000 for 2 years. The nominal interest rate is 8%. How much will we have at the end of the second year?

Answer:

2 years is 8 quarters.

The first deposit will be capitalised 8 times, the next 7, the next 6, ... and the last deposit 1 time.

So after 2 years we will have:

$$1\ 000(1+0.02)^8 + 1\ 000(1+0.02)^7 + 1\ 000(1+0.02)^6$$
$$+1\ 000(1+0.02)^5 + 1\ 000(1+0.02)^4 + 1\ 000(1+0.02)^3$$
$$+1\ 000(1+0.02)^2 + 1\ 000(1+0.02)^1 = 8\ 754.63\ \text{PLN}$$

What if we paid PLN 1 000 for 3 years every month?

We would need to calculate the sum of 36 components.

Is there any shorter way?

This stream of payments in financial mathematics is called ANNUITY.

Annuity – a series of identical deposits or withdrawals, made every certain fixed, specified period of time, at the same interest rate.

The value of deposits or withdrawals is called the amount of money stream or payment, denoted by PMT (Payment).

The general formula for rent can be determined using the sum of a geometric sequence.

It should be mentioned that there are 2 types of annuity:

- Retirement pension payable in arrears / ordinary annuity – payments occur at the end of each period.
- Annuity due – payments occur at the beginning of each period.

For each of these annuities, we can calculate a future value (FVA) and a present value (PVA).

2.1. Future Value of Annuity

Future Value of Annuity (FVA) / for annuity in advance – FVAD / – such amount of capital at the end of the last period that is equivalent to a series of payments at a given interest rate and for a given number of periods.
How to derive the formula?

2.1.1. Annuity in arrears (end-of-period payments)

Suppose we have PMT payments that occur at the end of each of n periods. The nominal interest rate is r.

First payment is made at the end of the first period, hence it will be capitalised (n-1) times, another (n-2) times etc., and the last one is made at the end of the last period so it won’t be capitalised at all

$$FVA = PMT(1+r)^{n-1} + PMT(1+r)^{n-2} + PMT(1+r)^{n-3} + \cdots$$

$$+ PMT(1+r) + PMT$$

$$= PMT((1+r)^{n-1} + (1+r)^{n-2} + (1+r)^{n-3} + \cdots + (1+r) + 1)$$

$$= PMT(1 + (1+r) + \cdots + (1+r)^{n-3} + (1+r)^{n-2} + (1+r)^{n-1})$$

In parentheses we have the sum of n terms of the geometric sequence in which $q = (1+r)$ and $a_1 = 1$, that's why we use the formula $S_n = a_1 \frac{1-q^n}{1-q}$ and we have:

$$FVA = PMT \cdot 1 \cdot \frac{1-(1+r)^n}{1-(1+r)} = PMT \cdot \frac{1-(1+r)^n}{-r} = PMT \cdot \frac{(1+r)^n - 1}{r}$$

$$FVAM(n,r) = \frac{(1+r)^n - 1}{r}$$

$FVAM(n,r)$ is Future value of annuity multiplier.

2.1.2. Annuity in advance (payments at the beginning of the period)

Suppose we have PMT payments that occur at the beginning of each of n periods. The nominal interest rate is r.

First payment is made at the beginning of the first period, hence it will be capitalised (n) times, another (n-1) times, etc., and the last one is made at the beginning of the last period, so it will be capitalised once

$$FVAD = PMT(1+r)^n + PMT(1+r)^{n-1} + PMT(1+r)^{n-2} + \cdots$$

$$+PMT(1+r)$$

$$= PMT\big((1+r)^n + (1+r)^{n-1} + (1+r)^{n-2} + \cdots + (1+r)\big) =$$

We exclude before the bracket (1 + r) and we get the same expression in brackets as for the annuity in arrears

$$PMT(1+r)(1 + (1+r) + \cdots + (1+r)^{n-3} + (1+r)^{n-2} + (1+r)^{n-1})$$

In parentheses we have the sum of n terms of the geometric sequence,
where q = (1 + r) and $a_1 = 1$, that's why we use the Formula $S_n = a_1 \frac{1-q^n}{1-q}$ and we have:

$$FVAD = PMT \cdot (1+r) \cdot 1 -$$

$$FVAM(n,r) = \frac{(1+r)^n - 1}{r}$$

$FVAM(n,r)$ is Future value of annuity multiplier as previously shown.

2.2. Present Value of Annuity

Present Value of Annuity (PVA) / for annuity in advance PVAD / – such amount of capital at time t = 0 that is equivalent to a series of payments at a given interest rate and for a given number of periods.

2.2.1. Annuity in arrears (end-of-period payments)

Suppose we have PMT payments that occur at the end of each of n periods. The nominal interest rate is r.

We calculate the present value for the present moment (i.e. moment 0)

The first payment is made at the end of the first period hence it will be discounted 1 time, the next one 2 times etc., and the last one is made at the end of the last period, so it will be discounted (n) times.

$$PVA = \frac{PMT}{1+r} + \frac{PMT}{(1+r)^2} + \cdots + \frac{PMT}{(1+r)^{n-1}} + \frac{PMT}{(1+r)^n} =$$

$$PMT\left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{n-1}} + \frac{1}{(1+r)^n}\right) =$$

In parentheses we have the sum of n terms of the geometric sequence, where

$q = \frac{1}{1+r}$ And $a_1 = \frac{1}{1+r}$, so we use the formula$S_n = a_1 \frac{1-q^n}{1-q}$ and we have:

$$PVA = PMT \cdot \frac{1}{1+r} \cdot \frac{1-\left(\frac{1}{1+r}\right)^n}{1-\frac{1}{1+r}} = PMT \cdot \frac{1}{1+r} \cdot \frac{1-\left(\frac{1}{1+r}\right)^n}{\frac{1+r}{1+r} - \frac{1}{1+r}} =$$

$$PMT \cdot \frac{1}{1+r} \cdot \frac{1-\left(\frac{1}{1+r}\right)^n}{\frac{r}{1+r}} = PMT \cdot \frac{1}{1+r} \cdot \left(1-\left(\frac{1}{1+r}\right)^n\right) \cdot \frac{1+r}{r} =$$

$$PMT \cdot \frac{1-\left(\frac{1}{1+r}\right)^n}{r}$$

$$PVAM(n,r) = \frac{1-\left(\frac{1}{1+r}\right)^n}{r}$$

$PVAM(n,r)$ is Present value of annuity multiplier

2.2.2. Annuity in advance (payments at the beginning of the period)

Suppose we have PMT payments that occur at the beginning of each of n periods. The nominal interest rate is r.

We calculate the present value for the present moment (i.e. moment 0)

The first payment is made at the beginning of the first period, hence it will not be discounted, the next one will be discounted 1 time etc., and the last one is made at the beginning of the last period, so it will be discounted (n-1) times.

$$PVAD = PMT + \frac{PMT}{1+r} + \frac{PMT}{(1+r)^2} + \cdots + \frac{PMT}{(1+r)^{n-2}} + \frac{PMT}{(1+r)^{n-1}}$$

$$PMT\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^{n-2}} + \frac{1}{(1+r)^{n-1}}\right) =$$

$$PMT\cdot(1+r)\left(\frac{1}{1+r}+\frac{1}{(1+r)^2}+\cdots+\frac{1}{(1+r)^{n-1}}+\frac{1}{(1+r)^n}\right)$$

In parentheses we have the sum of n terms of the geometric sequence, where $q=\frac{1}{1+r}$ And $a_1=\frac{1}{1+r}$, so we use the formula$S_n=a_1\frac{1-q^n}{1-q}$ and we have:

$$PVAD=PMT\cdot(1+r)\cdot\frac{1}{1+r}\cdot\frac{1-\left(\frac{1}{1+r}\right)^n}{1-\frac{1}{1+r}}=PMT\cdot(1+r)\cdot\frac{1}{1+r}\cdot\frac{1-\left(\frac{1}{1+r}\right)^n}{\frac{1+r}{1+r}-\frac{1}{1+r}}=$$

$$PMT\cdot(1+r)\cdot\frac{1}{1+r}\cdot\frac{1-\left(\frac{1}{1+r}\right)^n}{\frac{r}{1+r}}=PMT\cdot(1+r)\cdot\frac{1}{1+r}\cdot\left(1-\left(\frac{1}{1+r}\right)^n\right)\cdot\frac{1+r}{r}=$$

$$PMT\cdot\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\cdot(1+r)$$

$$PVAM(n,r)=\frac{1-\left(\frac{1}{1+r}\right)^n}{r}$$

$PVAM(n,r)$ is Present value of annuity multiplier as previously shown.

	Ordinary annuity	Annuity due
Future value	$FVA=PMT\frac{(1+r)^n-1}{r}$ or $FVA=PMT\cdot FVAM(n,r)$	$FVAD=PMT\frac{(1+r)^n-1}{r}\cdot(1+r)$ or $FVAD=PMT\cdot FVAM(n,r)\cdot(1+r)$
Present value	$PVA=PMT\frac{1-\frac{1}{(1+r)^n}}{r}$ or $PVA=PMT\cdot PVAM(n,r)$	$PVAD=PMT\frac{1-\frac{1}{(1+r)^n}}{r}(1+r)$ or $PVAD=PMT\cdot PVAM(n,r)\cdot(1+r)$

2.3. Perpetual annuity

Perpetual annuity – an annuity (ordinary) with payments that never end, and therefore with an infinite number of payments. Only its present value can be calculated, its future value cannot be determined, because the ordinary annuity has no limit for n → ∞.

For an annuity in arrears

$$PVA^{\infty}=\frac{PMT}{1+r}+\frac{PMT}{(1+r)^2}+\cdots=PMT\left(\frac{1}{1+r}+\frac{1}{(1+r)^2}+\cdots\right)=$$

In parentheses we have the sum of the infinite geometric sequence in which $a_1=\frac{1}{1+r}$ and $q=\frac{1}{1+r}$, so:

$$PVA^{\infty} = PMT \cdot \frac{\frac{1}{1+r}}{1-\frac{1}{1+r}} = PMT \cdot \frac{1}{1+r} \cdot \frac{1+r}{1+r-1}$$

$$= PMT \cdot \frac{1}{1+r} \cdot \frac{1+r}{r} = \frac{PMT}{r}$$

For annuity in advance

$$PVAD^{\infty} = PMT + \frac{PMT}{1+r} + \frac{PMT}{(1+r)^2} + \cdots = PMT\left(1 + \frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots\right) =$$

In parentheses we have the sum of the infinite geometric sequence in which $a_1 = 1$ and $q = \frac{1}{1+r}$, so

$$PVAD^{\infty} = PMT \cdot \frac{1}{1-\frac{1}{1+r}} = PMT \cdot 1 \cdot \frac{1+r}{1+r-1} = PMT \cdot \frac{1+r}{r} = \frac{PMT}{r} \cdot (1+r)$$

Example 1.

We make payments of PLN 1 000 every month for 3 years, with a nominal rate of 12%. What amount will we have after 3 years if the payments are:
a) At the beginning of the month,
b) At the end of the month.

We have:

$$PMT = 1\ 000, m = 12, n = 3$$

We have 36 payments, and the monthly rate is 1%.

a) $FVAD = 1\ 000 \cdot FVAM(36.1\%)(1 + 0.01) = 43\ 507.65$

b) $FVA = 1\ 000 \cdot FVAM(36.1\%) = 43\ 076.68$

Example 2.

What amount do you need to have today to be able to withdraw PLN 1 000 every month for 3 years, at a nominal rate of 12%, if the payments are made:
a) At the beginning of the month,
b) At the end of the month.

We have:

$$PMT = 1\ 000, m = 12, n = 3$$

We have 36 payments, and the monthly rate is 1%.

a) $PVAD = 1\ 000 \cdot PVAM(36.1\%)(1 + 0{,}01) = 30\ 408.58$

b) $PVA = 1\ 000 \cdot PVAM(36.1\%) = 30\ 107.51$

Example 3.

What amount do you need to have today to be able to withdraw PLN 1 000 every month for life, with a nominal rate of 12%, if the payments are made:
a) At the beginning of the month,
b) At the end of the month.

So we have a perpetual annuity

a) $PVAD^{\infty} = \frac{1\ 000}{0.01} \cdot 1.01 = 101\ 000$

b) $PVA^{\infty} = \frac{1\ 000}{0.01} = 100\ 000$

3. Loans – debt repayment plans

Symbols:

$A = [S, -A_1, -A_2, \dots, , -A_n]$ – the stream of cash flows in moments
$n = 1,2, \dots, N$.
S – debt incurred at time 0,
$A_1, A_2, \dots, A_n$ – installments (payments) , $n = 1,2, \dots, N$.
S_n – debt immediately after making a payment A_n.
K_n – capital part of the nth installment,
T_n – interest part of the nth installment

$$A_n = K_n + T_n$$

r – interest rate, nominal interest rate on the loan,
N – number of installments

Dependencies:

The following condition must be met:

$$S = \sum_{i=1}^{N} \frac{A_i}{(1+r)^i}$$

The value of the loan must be equal to the sum of the discounted payments (installments), therefore we can look at the loan as the present value of the annuity, the payments of which are equal to the loan installments.

The debt after the nth repayment can be calculated using the following formulas:

$$S_n = S(1+r)^n - \sum_{i=1}^{n} \frac{A_i}{(1+r)^{n-i}}$$

$$S_n = \sum_{i=n+1}^{N} \frac{A_i}{(1+r)^{n-i}}$$

3.1. Debt repayment scheme

When creating a debt repayment plan, remember that the capital installments sum up to the loan amount, and we always count interest on the current debt, and the payment is the sum of the principal installment and interest. By applying these rules, we can create a repayment plan for any loan.

Example 1.

Prepare a debt repayment plan of PLN 1 000 to be paid back in 4 quarterly installments, with a nominal rate of r = 12%, if it is known that the principal installments are as follows: 300, 200, 400, 100.

(We make payments quarterly, so the quarterly interest rate is 3%)

n	Debt at the beginning of period n	Interest part of installment	Installment	Capital part of installment	Debt at the end of the period
1	1000	$T_1 = 0.03 \cdot 1000 = 30$	$A_1 = 300 + 30 = 330$	300	$S_1 = 1000 - 300 = 700$
2	700	$T_2 = 0.03 \cdot 700 = 21$	$A_2 = 200 + 21 = 221$	200	$S_2 = 700 - 200 = 500$
3	500	$T_3 = 0.03 \cdot 500 = 15$	$A_3 = 400 + 15 = 415$	400	$S_3 = 500 - 400 = 100$
4	100	$T_4 = 0.03 \cdot 100 = 3$	$A_4 = 100 + 3 = 103$	100	$S_4 = 400 - 400 = 0$
Total		69	1069	1000	

3.1.1. Equal capital installments (decreasing installments, annuity installments)

We repay the same part of the capital in each installment. We calculate interest on the current debt, therefore, as the debt decreases, the interest is also decreasing, which causes the installments to decrease.

$$K_1 = K_2 = \cdots = K_N = \frac{S}{N}$$

$$T_n = r \cdot S_{n-1} = S \cdot r \frac{N - (n-1)}{N}$$

$$A_n = K_n + T_n$$

3.1.2. Equal payments (the sum of the principal and interest installments is fixed)

In this model $A_1 = A_2 = \cdots = A_N$, thus

$$S = \sum_{i=1}^{N} \frac{A_i}{(1+r)^i} = A \sum_{i=1}^{N} \frac{1}{(1+r)^i} = A \cdot PVAM(r, N)$$

And

$$A = \frac{S}{PVAM(r, N)}$$

Example 2.

Prepare a debt repayment plan of PLN 2 000, to be repaid in four equal six-month capital installments. Nominal interest rate r = 8%.

n	Debt at the beginning of period n	Interest part of installment	Installment	Capital part of installment	Debt at the end of the period
1	2000	$T_1 = 0.04 \cdot 2000 = 80$	$A_1 = 500 + 80 = 580$	500	$S_1 = 2000 - 500 = 1500$
2	1500	$T_2 = 0.04 \cdot 1500 = 60$	$A_2 = 500 + 60 = 560$	500	$S_2 = 1500 - 500 = 1000$
3	1000	$T_3 = 0.04 \cdot 1000 = 40$	$A_3 = 500 + 40 = 540$	500	$S_3 = 1000 - 500 = 500$
4	500	$T_4 = 0.04 \cdot 500 = 20$	$A_4 = 500 + 20 = 520$	500	$S_4 = 500 - 500 = 0$
Total		200	2200	2000	

Example 3

The loan of PLN 10 000 is to be repaid in 8 equal annual capital installments. Nominal interest rate of 5%. Calculate:

a) Amount of the capital installment,

b) Debt after the third installment,

c) Interest on the fifth payment,

d) Amount of the seventh payment.

Answer:

a) $K = \frac{10\ 000}{8} = 1\ 250$

b) We have paid three installments, therefore $S_3 = 10\ 000 - 3 \cdot 1\ 250 = 6\ 250$

c) We have paid four installments, so our debt equals

$$S_4 = 10\ 000 - 4 \cdot 1250 = 5\ 000$$

We calculate interest on the current debt, so

$$T_5 = 0.05 \cdot 5\ 000 = 250$$

d) We have paid six installments, so our debt equals

$$S_6 = 10\ 000 - 6 \cdot 1250 = 2\ 500$$

Interest $T_7 = 0.05 \cdot 2\ 500 = 125$

So the seventh payment is $A_7 = 1\ 250 + 125$ = 1 375.

Example 4

The loan of PLN 10 000 is to be repaid in 8 equal half-year payments. Nominal interest rate is 4%. Calculate:

a) Installment (payment) amount,

a) Debt after the fourth installment,

b) Interest on the fifth payment,

c) Capital installment in the fifth payment.

Answer:

$$A = \frac{10\ 000}{PVAM(8.2\%)} = \frac{10\ 000}{7.3255} = 1\ 365.10$$

We have paid four installments, so we use the formula

$$S_n = S(1+r)^n - \sum_{i=1}^{n} \frac{A_i}{(1+r)^{n-i}}$$

Because we have equal payments, thus

$$S_n = S(1+r)^n - A\sum_{i=1}^{n} \frac{1}{(1+r)^{n-i}} = S \cdot FVM(n,r) - A \cdot FVAM(n,r)$$

$$S_4 = 1\ 000 \cdot FVM(4.2\%) - 1\ 365.1 \cdot FVAM(4.2\%) =$$

$$10\ 000 \cdot 1.0824 - 1\ 365.1 \cdot 5.2040 = 3\ 936.86$$

Because
$S_4 = 3\ 936.86$, więc $T_5 = 0.02 \cdot 3\ 936.86 = 78.74$

$$K_5 = A - T_5 = 1\ 365.1 - 78.74 = 1\ 286.4$$

4. Valuation of debt instruments

Debt instruments – securities that are evidence of a financial obligation between the debtor and the buyer of the debt security. The debtor undertakes to return the entire amount within the specified period, together with interest on the use of the capital. The instruments are usually highly liquid and have little investment risk. They are therefore used to obtain funds to finance current operations. Moreover, they are also a form of investing free capital, which brings profit in the form of interest.

Types of debt instruments:

- bonds,
- Treasury bills,
- Certificates.

4.1. Bonds

A bond is a security in which the bond issuer, i.e. the borrower, confirms taking a specific loan amount and undertakes to return it to the bond buyer (lender) within the prescribed period and to pay interest. For the issuer, it is an instrument with which it can borrow from many creditors at the same time, often for a very long period, even considerably longer than the period for which banks specialising in this field are willing to grant investment loans.

For an investor, bonds are a very simplified form of investing capital, calculated for a stable, predetermined income, regardless of the benefits the issuer obtains thanks to the loan. With the help of bonds, the issuer can obtain even large amounts of loans consisting of the savings of many investors.

Due to the characteristics of the income generated by the bonds, we can distinguish:

- fixed rate bonds that bring you a fixed income; its interest rate is fixed and known, which makes it possible to determine all payments for holding the bond at the time of purchase; they account for about 2/3 of the total bond issue in the world,
- floating-rate bonds that bring variable income related to the interest rate that is verified from time to time,
- zero-coupon bonds, where the only benefit from the issuer occurs at redemption, i.e. the issuer pays no interest but sells the bonds at a price lower than the redemption price,

A bond is a debt security and the basic issue considered in the analysis of debt instruments is their valuation, that is, determining their value. The value of the debt instrument determined as a result of the valuation is compared with the market price of this instrument. In the course of the valuation, the result may indicate three different cases:

- undervaluation, underpricing – the value is higher than the market price, which means a signal to buy this instrument,
- overvalued, overpricing – the value is lower than the market price, which means a signal to sell this instrument,
- well priced instrument – the value is equal to the market price.

4.1.1. Valuation

The market value of a bond, taking into account the time value of money, when held to maturity and when interest is fully reinvested, is calculated using the Yield to Maturity calculation:

$$P = \sum_{t=1}^{n} \frac{C_t}{(1+YTM)^t} + \frac{W_N}{(1+YTM)^n} = C \cdot \frac{1 - \frac{1}{(1+YTM)^n}}{d} + \frac{W_N}{(1+YTM)^n} =$$

$$P = C \cdot PVAM(n,r) + W_n \cdot PVM(n,r)$$

where:
P – bond value,
YTM – yield to maturity,
C_t – interest in year t,
W_N – nominal value.

An investment in bonds does not bring the same income as an investment in stocks, but the advantage of the former is that the risk of incurring a loss is much lower. In the case of government bonds, it is almost zero, because the state does not go bankrupt. A well-known saying in the financial market says: stock investors eat better, but the bondholder sleeps more peacefully.

Government bonds are widely recognised as the most secure securities. Through them, the state budget borrows money from citizens, which it undertakes to return with interest in a timely and predetermined manner. The business is mutually beneficial. In this way, the state obtains the necessary funds to finance budgetary expenses, while creditors, i.e. bondholders, receive a guaranteed income.

4.1.2. Duration

Duration – the duration or average maturity of the bond. It is used to determine the risk of a price change. The lower the interest rate, the greater the risk of a price change. The longer the maturity date, the greater the risk of price changes. Thus, duration is a measure of the risk of a price change and allows you to compare bonds with different maturities and different interest rates.

Bond's duration, including periodicity – weighted average period of waiting for the inflow of cash from bonds (average maturity of bonds). This time is a measure of the sensitivity of changes in the bond price to changes in market interest rates. In other words, it allows for the ordering of bonds according to the interest rate risk. Among the bonds with the same maturity, the highest interest rate risk is that of zero-coupon bonds, as their duration is equal to the maturity.

Knowing the bond duration it can be used for:

- hedging against interest rate risk;
- estimating price changes with changes in profitability in order to hedge against an operation with the same risk, but made in the opposite direction;
- matching maturity of assets and liabilities;
- estimating price changes with specific profitability in trading;
- use in portfolio management.

The duration of a bond can be determined from the formula:

$$D = \frac{1}{P}\sum_{t=1}^{n}\frac{t \cdot C_t}{(1+YTM)^t}$$

4.1.3. Convexity of bonds

In finance, the convexity of bonds is a measure of the non-linearity of changes in the price of a bond depending on interest rates.
For bonds with a coupon, the convexity is calculated using the following formula:

$$C = 0.5\frac{\sum_{t=1}^{n}\frac{t(t+1) \cdot C_t}{(1+YTM)^t}}{P \cdot (1+YTM)^2}$$

From the above formula, the following conclusions can be drawn:

- The higher the interest rate, the smaller the convexity of the bond (with equal yields and equal maturity).
- The greater the t, the greater the convexity.
- The greater the YTM, the smaller the convexity.

Convexity formula for a zero-coupon bond:

$$C = \frac{0.5 \cdot n(n+1)}{(1+YTM)^2}$$

n – number of years to maturity

4.2. Certificate of deposit

Certificate of deposit – a transferable security issued by a bank in order to accumulate funds, certifying that the bearer of the certificate of deposit has deposited a certain amount of funds with the bank for a specified period of time, after which the issuing bank is obliged to return this amount together with the specified interest.

In Poland, the regulations of issuing certificates of deposit are set out in Articles 89-92 of the Banking Law, where the certificate of deposit is defined as a bank security used by banks to accumulate funds in zlotys or in another convertible currency.

Certificates of deposit issued by commercial banks and cooperative banks, both as short-term instruments (with maturity up to 1 year) and long-term instruments (with maturity up to 5 years), can take both material and dematerialised forms. The sale of certificates of deposit on the primary market is usually conducted through a bank other than the issuer in order to increase the range of distribution of these securities.

They are usually issued in rounded large amounts, such as €100 000. Their interest rate is higher than that of some securities and they can be sold on the secondary market before maturity, which makes them willingly purchased by enterprises.
Formula for a simple rate of income for a certificate of deposit:

$$r = \frac{\left[\frac{FV\left(1 + \frac{i \cdot N_{\Im}}{360}\right)}{P} - 1\right] \cdot 360}{N_{pm}}$$

r – instrument yield rate, profitability rate,

FV – the nominal value,

P – price,

N_{im} – the number of days between the issue date and the maturity date,

Nsm – the number of days between the sale date and the maturity date (when the investor does not hold the instrument until the maturity date),

Npm – the number of days between the purchase date and the maturity date.

This is the rate without income reinvestment, that is, without capitalisation.

If we want to apply reinvestment of income (capitalisation), the following formula for the effective rate of income should be used:

$$r = \left[\frac{FV\left(1 + \frac{i \cdot N_{\Im}}{360}\right)}{P}\right]^{\frac{360}{N_{pm}}} - 1$$

We determine the value of the deposit based on the following formula:

$$P = \frac{FV\left(1 + \frac{i \cdot N_{\Im}}{360}\right)}{1 + r\frac{N_{pm}}{360}}$$

When an investor buys a certificate of deposit after the issue date and sells before the maturity date, the yield is calculated on the basis of the formula

$$r = \frac{\left[\dfrac{1 + r_p \dfrac{N_{pm}}{360}}{1 + r_s \dfrac{N_{sm}}{360}} - 1\right] \cdot 360}{N_{pm} - N_{sm}}$$

r_p – profitability rate of the certificate of deposit on the date of purchase,

r_s – the profitability rate of the certificate of deposit on the date of sale.

4.3. Treasury Bill

Treasury security – short-term debt security issued by the government.

The basic feature of Treasury Bills is a very low degree of risk, which makes them an attractive investment paper. For zero risk, the buyer pays the cost in the form of a low interest rate.

In Poland, Treasury Bills are issued by the Minister of Finance. The bills are bearer securities with a nominal value of PLN 10 000, issued for a period of 1 to 90 days or 1 to 52 weeks. The interest rate on the bills is fixed. The coupons are discounted (the income is the difference between the purchase price and the nominal value of the voucher).

When the investor holds the Treasury Bill until maturity, the discount rate is:

$$d = \frac{\left(1 - \dfrac{P}{FV}\right) \cdot 360}{N_{pm}}$$

Npm – the number of days between the date of purchase and the date of redemption of the Treasury Bill,

P – price,

FV – nominal value,

d – discount rate.

The profitability rate is calculated on the basis of the formula:

$$r = \frac{\left(\dfrac{FV}{P} - 1\right) \cdot 360}{N_{pm}}$$

The value of Treasury Bills is determined on the basis of the following formula:

$$P = \frac{FV}{1 + \dfrac{r \cdot N_{pm}}{360}}$$

The formula for the effective rate of a Treasury Bill with complex capitalisation:

$$r = \left(\frac{FV}{P}\right)^{\frac{360}{N_{pm}}} - 1$$

When an investor buys Treasury Bills after the issue date and sells before the maturity date, we calculate the yield using the formula:

$$r = \left(\frac{1 + r_p \frac{N_{pm}}{360}}{1 + r_s \frac{N_{sm}}{360}} - 1 \right) \cdot \left(\frac{360}{N_{pm} - N_{sm}} \right)$$

Sample tasks:

1. How much should you invest today to receive a monthly annuity of PLN 2 000, assuming that the monthly discount rate is 2%, where:

   a) Payments would be made at the end of the periods,

   b) Payments would be made at the beginning of the periods.

2. PLN 18 000 had been paid into the bank. After 5 years an additional amount PLN 3 000 was paid at the beginning of each year. What capital will be created in this way after ten years? The bank applies compound annual capitalisation at an annual interest rate of 2%.
3. The nominal interest rate is 10%. Determine the future value of payments of PLN 100 after 10 years, if – with annual capitalisation of interest – they were paid:

   a) at the end of each year,

   b) at the beginning of each year.

4. For 20 years, at the beginning of each year, a fixed amount of PLN 4 000 had been set aside to a certain fund. After 30 years, the fund began to pay out a perpetual annuity consisting of constant annual payments. Find the amount of those fixed annual payments, knowing that the rate was 10% for the entire period and the first payment was made at the end of the 30th year.
5. Prepare a loan amortisation plan in the amount of PLN 1 000 if it is repaid in four equal principal installments payable at the end of each quarter and the nominal interest rate is 20%. Interest capitalisation is quarterly.
6. Prepare a loan amortisation plan in the amount of PLN 12 000, if its repayment is made in four equal amounts of payments made at the end of each quarter, and the nominal interest rate is 20%. Interest capitalisation is quarterly.
7. A loan of PLN 10 000 has an interest rate of 20% per year. The loan is to be repaid in equal payments over 4 years at the end of each quarter.

   a) What is the amount of the quarterly payment amount?

   b) What is the amount of the loan after paying the tenth installment?

   c) How much is the interest payable in the eleventh quarter?

8. A loan of PLN 5 000 is repaid in 10 equal principal installments payable at the end of each month. The nominal interest rate is 12%, the capitalisation is monthly.

   a) Calculate the loan amount after 6 months.

   b) How much is the interest for the 4th month?

   c) What will be the payment amount to be made in the 3rd month?

Examples of questions:

- What is an annuity?
- Types of loans.
- What are Treasury Bills and who is their issuer?

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